Partially Directed nil-Temperley-Lieb Algebras

Maya Sankar
mentor: Dr. Tanya Khovanova
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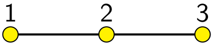
nil-Temperley-Lieb (nTL) Algebras



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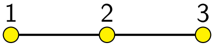

- Algebra based on a graph G. One generator per vertex: x_1, x_2, x_3.
- $x_i^2 = 0$.
- For two adjacent vertices i and j, $x_i x_j x_i = x_j x_i x_j = 0$.
- For two nonadjacent vertices i and j, $x_i x_j = x_j x_i$.
- A monomial that does not equal 0 is called irreducible.

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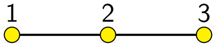

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Example

$x_3 x_1 x_2 x_3 = x_1 x_3 x_2 x_3 = 0$ is reducible.

$x_2 x_1 x_3 x_2 = x_2 x_3 x_1 x_2$ is irreducible.

Dimension of the Algebra

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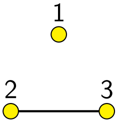
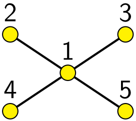
G_1:



G_2:



In G_1, these monomials are

$$1, \quad x_1,\ x_2,\ x_3, \quad x_1x_2,\ x_1x_3,\ x_2x_3,\ x_3x_2, \quad x_1x_2x_3,\ x_1x_3x_2$$

and the dimension is 10. Not counted are repeated monomials ($x_2x_1 = x_1x_2$ and $x_3x_1 = x_1x_3$) and reducible monomials ($x_2x_3x_2 = 0$ and $x_3x_2x_3 = 0$).

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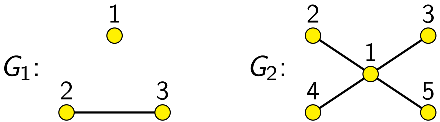

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In G_2, there is an infinite irreducible monomial:

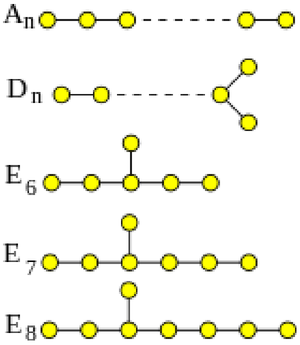
$$x_1 x_2 x_3 x_1 x_4 x_5 x_1 x_2 x_3 x_1 x_4 x_5 \ldots$$

$$= x_1 x_3 x_2 x_1 x_5 x_4 x_1 x_3 x_2 x_1 x_5 x_4 \ldots$$

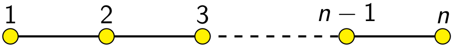
Dimension of the Algebra

Theorem

The nTL algebra on G is finite iff G is a Dynkin diagram.



nil-Temperley-Lieb Algebras on the Path Graph



- Number the vertices 1 to n.
- Dimension of the algebra known to be C_{n+1}, the $n+1^{\text{th}}$ Catalan number.

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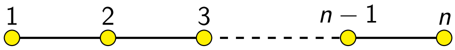

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- Dimension of the algebra known to be C_{n+1}, the $n+1^{\text{th}}$ Catalan number.
- Each monomial can be uniquely written as a series of decreasing runs, with increasing peaks and valleys.
  $(x_3 x_2 x_1)(x_5 x_4 x_3 x_2)(x_7 x_6)$
  - This is the lexicographically smallest representation of the monomial.

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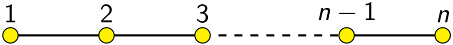

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  $(x_3 x_2 x_1)(x_5 x_4 x_3 x_2)(x_7 x_6)$
  - This is the lexicographically smallest representation of the monomial.
- If peaks don't increase:
  $x_4 x_3 x_2 x_1 x_4 x_3 = x_4 x_3 x_2 x_4 x_1 x_3 = x_4 x_3 x_4 x_2 x_1 x_3 = 0$

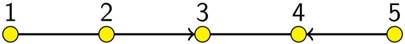
Motivation

- Map to the set of permutations on $n+1$ elements: if x_i is taken to the transposition of the i^{th} and $i+1^{\text{th}}$ elements.
  - By this construction, the elements of the algebra are 321-avoiding permutations.

Motivation

- Map to the set of permutations on $n+1$ elements: if x_i is taken to the transposition of the i^{th} and $i+1^{\text{th}}$ elements.
  - By this construction, the elements of the algebra are 321-avoiding permutations.
- Definitions similar to those of Coxeter groups. The elements of the algebra correspond to elements of Coxeter groups satisfying certain properties.

Partially Directed nTL Algebras



- Based on a graph G with some directed and some undirected edges.
- $x_i^2 = 0$.
- For two nonadjacent vertices i and j, $x_i x_j = x_j x_i$.
- For two vertices i and j connected by an undirected edge, $x_i x_j x_i = x_j x_i x_j = 0$.

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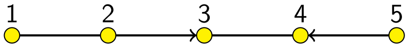

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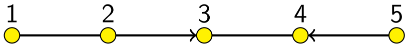

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- **For two vertices i and j with a directed edge from i to j, $x_i x_j x_i = 0$.**

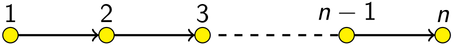
The example has relations $x_2 x_3 x_2 = 0$ and $x_5 x_4 x_5 = 0$, but not $x_3 x_2 x_3 = 0$ or $x_4 x_5 x_4 = 0$.

Dimensions of Partially Directed nTL algebras

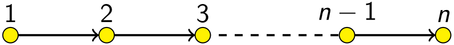
Theorem

The nTL algebra on a partially directed graph G is finite iff G is a path graph with all directed edges going in the same direction.

Maximally Directed nTL Algebras



Maximally Directed nTL Algebras



Each monomial can be written uniquely as a series of decreasing runs with increasing valleys. For example,

$$(x_5x_4x_3x_2x_1)\,(x_7x_6x_5x_4x_3)\,(x_6x_5x_4)\,(x_7)\,.$$

There are $n+1$ choices for the run with valley x_1:

$$1, \quad x_1, \quad x_2x_1, \quad \ldots, \quad x_nx_{n-1}\ldots x_2x_1.$$

Similarly, there are n choices for the run with valley x_2, $n-1$ choices for the run with valley x_3, and so on.

Maximally Directed nTL Algebras

Theorem

There are $(n+1) \times n \times (n-1) \times \ldots \times 2 = (n+1)!$ *elements in the maximally directed algebra.*

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Mapping the generator x_i to the transposition of i and $i+1$ in the set of permutations on $n+1$ elements, each irreducible monomial corresponds to a different element of the set of permutations on $n+1$ elements.

Peaks and Valleys

Every decreasing run has a peak and valley: $x_5 x_4 x_3 x_2 x_1$.

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Every partially directed nTL algebra is a subalgebra of the maximally directed nTL algebra. Thus,

Theorem

The monomials of a partially directed nTL algebra are sequences of decreasing runs with increasing valleys.

Conditions on the Peaks

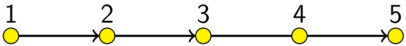
Theorem

If there is an undirected edge from i *to* $i+1$ *and there are two peaks with (from left to right)* $p_1 \geq i+1$ *and* $p_2 = i+1$*, there must be a peak of* i *between* p_1 *and* p_2*.*

Conditions on the Peaks

Theorem

If there is an undirected edge from i to $i+1$ and there are two peaks with (from left to right) $p_1 \geq i+1$ and $p_2 = i+1$, there must be a peak of i between p_1 and p_2.



For example, when there is an undirected edge between 3 and 4 ($i = 3$), $x_5 x_4 x_3 x_2 x_1 x_3 x_2 x_4$ is irreducible, but $x_5 x_4 x_3 x_2 x_1 x_2 x_4$ is not.

This theorem completely describes the irreducible monomials in the partially directed nTL algebras.

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Corollary

There is no condition on the peaks of the maximally directed algebra.

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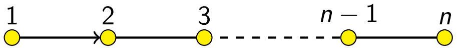
In the algebra based on the "undirected-directed" graph shown, peaks must strictly increase or remain higher than k.

Special Cases



Dimension: $C_n + C_{n+1} - 1$, where C_n is the n^{th} Catalan number.

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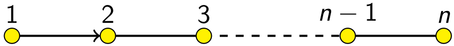

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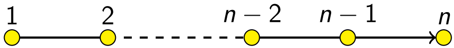

Dimension: $\binom{2n}{n} = (n+1)C_n$

Future research

- Find a general formula to calculate the dimension of *any* partially directed nTL algebra.

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- Further study which permutations are represented by a partially directed nTL algebra.

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- Further study which permutations are represented by a partially directed nTL algebra.
- A directed edge between i and j means changing the relation $x_i x_j x_i = x_j x_i x_j = 0$ to $x_i x_j x_i = 0$. What if we changed it to $x_i x_j x_i = x_j x_i x_j$?

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- Professor Postnikov for suggesting this project
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